Lecture 7

Tuesday, September 20, 2016 9:24 AM

- Rates of change

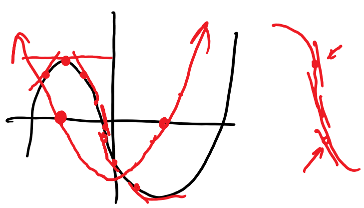
$y = f(x)$

If x changes from x_1 to x_2, denote the change by $\Delta x = x_2 - x_1$ and the corresponding change in y, $\Delta y = f(x_2) - f(x_1)$

The difference quotient

$\frac{\Delta y}{\Delta x} = \frac{f(x_2) - f(x_1)}{x_2 - x_1}$ is the average rate of change of y with respect to x over the interval $[x_1, x_2]$.

Consider the average rate of change over smaller and smaller intervals i.e. by letting x_2 get closer and closer to x_1 i.e. $\Delta x \to 0$.



DEF The limit of the average rate of change as $\Delta x \to 0$ is called the instantaneous rate of change of $y = f(x)$ at $x = x_1$

$$\lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{x_2 \to x_1} \frac{f(x_2) - f(x_1)}{x_2 - x_1}$$

$$= f'(x_1)$$

DEF The derivative $f'(a)$ can be interpreted as the instantaneous rate of change of $y = f(x)$ with respect to x at $x = a$.

Ch 2.8

Recall $f'(x) \overset{\text{DEF}}{=} \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$

Domain includes points in the domain of f where the above limit exists.

Ex If $f(x) = \sqrt{x}$, find $f'(x)$ and state it's domain.

- $$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h \to 0} \frac{\sqrt{x+h} - \sqrt{x}}{h}$$

$$= \lim_{h \to 0} \frac{\sqrt{x+h} - \sqrt{x}}{h} \cdot \frac{\sqrt{x+h} + \sqrt{x}}{\sqrt{x+h} + \sqrt{x}}$$

$$= \lim_{h \to 0} \frac{\cancel{x} + h - \cancel{x}}{h(\sqrt{x+h} + \sqrt{x})} \qquad (a+b)(a-b) = a^2 - b^2$$

$$= \lim_{h \to 0} \frac{\cancel{h}}{\cancel{h}(\sqrt{x+h} + \sqrt{x})} = \lim_{h \to 0} \frac{1}{\sqrt{x+h} + \sqrt{x}}$$

$$= \frac{1}{\sqrt{x} + \sqrt{x}} = \frac{1}{2\sqrt{x}}$$

$$f(x) = \sqrt{x}, \quad f'(x) = \frac{1}{2\sqrt{x}}$$

Domain of f : $x \geq 0$ $[0, \infty)$

Domain of f' : $x > 0$ $(0, \infty)$ //

OTHER NOTATIONS

$y = f(x)$

depend

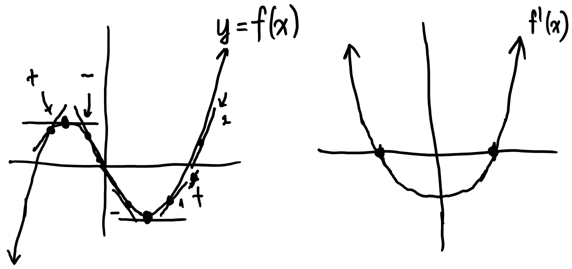
independent variable

$$f'(x) = y' = \frac{dy}{dx} = \frac{df}{dx} = \frac{d}{dx}(f(x))$$

$$f'(a) = \left.\frac{dy}{dx}\right|_{x=a}$$

Ex $y = f(x)$ $f'(x)$

Ex



Differentiable functions

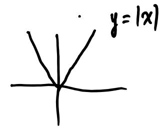
A function f is differentiable at a if $f'(a)$ exists.

$$f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$

A function is diff on an open interval (a,b) if it is diff at every number in the interval.

Ex Is $f(x) = |x|$ differentiable at $x = 0$?

$$f(x) = |x| = \begin{cases} x, & x \geq 0 \\ -x, & x < 0 \end{cases}$$



Does $\lim\limits_{x \to 0} \dfrac{f(x) - f(0)}{x - 0}$ exist ?

$$= \lim_{x \to 0} \frac{|x| - |0|}{x - 0} = \lim_{x \to 0} \frac{|x|}{x}$$

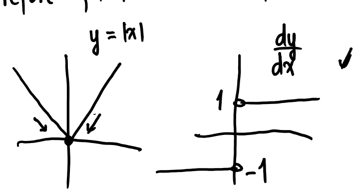
$$\lim_{x \to 0} \frac{|x|}{x} = ?$$

$$\lim_{x \to 0^+} \frac{|x|}{x} = \lim_{x \to 0^+} \frac{x}{x} = 1$$

$$\lim_{x \to 0^-} \frac{|x|}{x} = \lim_{x \to 0^-} \frac{-x}{x} = -1$$

$\lim\limits_{x \to 0} \dfrac{|x|}{x}$ DNE.

Therefore, $|x|$ is not diff at 0.



Rmk $f(x) = |x|$ is continuous $\left(\lim_{x \to a} f(x) = f(a)\right)$ at 0 but not diff at 0.

Thm If f is diff @ a, f is cont @ a.

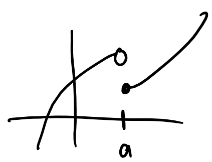
Refer to textbook for proof.

Q If f is not continuous at a,

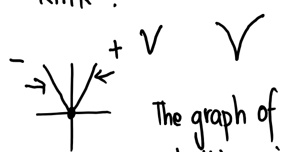
A then f is not diff @ a.

How can a function fail to be differentiable

1) f is not continuous at a, then f is not diff at a.



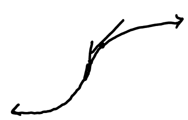
2) Graph of f has a "corner" or a "Kink".



The graph of f has no tangent at this point

3) Vertical Tangent Line

If The curve $y = f(x)$ has a vertical tangent line at $x = a$ (i.e $\lim_{x \to a^{\pm}} f'(x) = \pm\infty$) then it's not differentiable.



i.e tangent line gets steeper and steeper as $x \to a$.

Ex $f(x) = x^{1/3}$ @ $x = 0$.

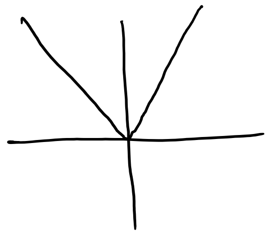
Ex $f(x) = x^{1/3}$ @ $x = 0$.

$$|x| = \begin{cases} x, & x \geq 0 \\ -x, & x < 0 \end{cases}$$

$$|\overset{x}{-0.3}| = 0.3$$

$$|x| = -x = -(-0.3) = 0.3$$

$$\lim_{x \to 0} \frac{|x|}{x} = \frac{0}{0}$$



$|-0.3| = 0.3$

$$f(x) = |x| = \begin{cases} \cancel{x, \; x \geq 0} \\ -x, & x < 0 \end{cases}$$

$-(-0.3) = 0.3$

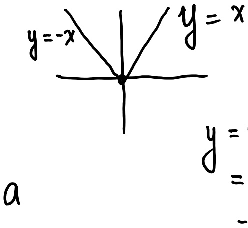
$y = -0.6$

$|\overset{y}{-0.6}| = 0.6$

$-y = -(-0.6)$
$= 0.6$

$-(-0.9) = 0.9$

$$f(x) = \underbrace{|x|} = \begin{cases} x, & x \geq 0 \\ \boxed{-x, \; x < 0} \end{cases}$$



$y = mx + b$
$= 1x + 0$
$= x$

$|0.9| = 0.9$ $\qquad f(a) = -a$

$f(\underset{a}{-0.9}) = |-0.9| = 0.9$

$$\lim_{x \to 0} \frac{|x|}{x} \qquad |x| = \begin{cases} x, & x \geq 0 \\ -x, & x < 0 \end{cases}$$

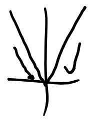
$\lim\limits_{x \to 0} \frac{|x|}{x}$ $|x| = \begin{cases} \\ -x, & x < 0 \end{cases}$

$\lim\limits_{x \to 0^+} \frac{|x|}{x} = \lim\limits_{x \to 0^+} \frac{x}{x} = 1$

$\lim\limits_{x \to 0} \frac{|x|}{x}$ DNE

$\lim\limits_{x \to 0^-} \frac{|x|}{x} = \lim\limits_{x \to 0^-} \frac{-x}{x} = -1$

$\lim\limits_{x \to 0} |x| = 0 = f(0)$



$0^- = 0$

Diff $\Rightarrow$ continuous

x

Not cont $\Rightarrow$ not diff

cont $\Rightarrow$ diff or not diff

$|x|$ 0

$|x|$